

NULL SPACES OF RADON TRANSFORMS

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ABSTRACT. We obtain new descriptions of the null spaces of several projectively equivalent transforms in integral geometry. The paper deals with the hyperplane Radon transform, the totally geodesic transforms on the sphere and the hyperbolic space, the spherical slice transform, and the Cormack-Quinto spherical mean transform for spheres through the origin. The consideration extends to the corresponding dual transforms and the relevant exterior/interior modifications. The method relies on new results for the Gegenbauer-Chebyshev integrals, which generalize Abel type fractional integrals on the positive half-line.

1. INTRODUCTION

In the present article we solve open problems stated in [22] and related to the structure of the kernels (null spaces) of several Radon-like transforms. These transforms are projectively equivalent to the hyperplane Radon transform of functions on \mathbb{R}^n and include the Funk transform on the sphere, its modification for spherical slices through the pole, the totally geodesic transform on the hyperbolic space, and the Cormack-Quinto transform, which integrates functions on \mathbb{R}^n over spheres passing through the origin.

All these transforms have unilateral structure. This important fact allows us to neglect some singularities, which restrict the corresponding classes of admissible functions. For example, in the case of the hyperplane Radon transform Rf on \mathbb{R}^n , we can exclude hyperplanes through the origin and consider only *almost all* hyperplanes in \mathbb{R}^n instead of *all* such planes. In such a setting, the behavior of f at the origin becomes irrelevant, and therefore, our consideration covers not only the classical Radon transform, but also its exterior version, when both f and Rf are supported away from a fixed ball.

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For square integrable functions, the kernels of the exterior Radon transform and the interior spherical mean transform were studied by Quinto [18, 19] who used the results of Ludwig [14]; see also Cormack and Quinto [7]. Our approach is different. It covers more general classes of functions and many other Radon-like transforms. For example, we show that for any fixed $a \geq 0$, the kernel of the Radon transform

$$(Rf)(\theta, t) = \int_{\theta^\perp} f(t\theta + u) d_\theta u, \quad \theta \in S^{n-1}, \quad |t| > a,$$

in the class of functions satisfying

$$\int_{|x| > a_1} \frac{|f(x)|}{|x|} dx < \infty \quad \forall a_1 > a, \quad (1.1)$$

is essentially the set of functions $\omega_{j,m}(x)$ of the form

$$\omega_{j,m}(x) = |x|^{2j-m-n} Y_m(x/|x|); \quad m=2, 3, \dots; \quad j=1, 2, \dots, [m/2],$$

where $[m/2]$ is the integer part of $m/2$, and Y_m is a spherical harmonic of degree m . The fact that these functions are annihilated by the operator R is not new; cf. [5, 6, 18, 19, 22]. The crucial point and one of the main results of the paper is that $\omega_{j,m}(x)$ *exhaust the kernel of R under the assumption (1.1)*; see Theorem 3.7 for the precise statement. Note that the assumption (1.1) is pretty weak in the sense that it is necessary for the existence of the Radon transforms of radial functions; see Theorem 3.1 below. Similar exact kernel descriptions are obtained for all Radon-like transforms mentioned above.

The paper is organized as follows. Section 2 is devoted to Gegenbauer-Chebyshev fractional integrals that form a background of our approach. In Section 3 we describe the kernels of the hyperplane Radon transform and its dual. The main results are stated in Theorems 3.7 and 3.6, which include the corresponding exterior and interior versions. Section 4 contains the description of the kernel of the Cormack-Quinto transform, which integrates functions on \mathbb{R}^n over spheres through the origin. Sections 5, 6, and 7 contain similar results for the Funk transform on the sphere, the spherical slice transform, and the totally geodesic transform on the hyperbolic space, respectively. Our assumptions for functions are inherited from (1.1) and provide the existence the corresponding Radon-like transforms in the Lebesgue sense.

Notation. As usual, $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the sets of all integers, positive integers, real numbers, and complex numbers, respectively; $\mathbb{Z}_+ = \{j \in \mathbb{Z} : j \geq 0\}$; $\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$. We will be dealing with the following function spaces:

$C(\mathbb{R}_+)$ is the space of continuous complex-valued functions on \mathbb{R}_+ ;
 $C_*(\mathbb{R}_+) = \{\varphi \in C(\mathbb{R}_+) : \lim_{t \rightarrow 0^+} \varphi(t) < \infty; \sup_{t > 0} t^k |\varphi(t)| < \infty \forall k \in \mathbb{Z}_+\};$
 $S(\mathbb{R}_+) = \{\varphi \in C^\infty(\mathbb{R}_+) : \varphi^{(\ell)} = (d/dt)^\ell \varphi \in C_*(\mathbb{R}_+) \forall \ell \in \mathbb{Z}_+\};$
 $C_c^\infty(\mathbb{R}_+) = \{\varphi \in C^\infty(\mathbb{R}_+) : \text{supp } \varphi \text{ is compact and } 0 \notin \text{supp } \varphi\}.$

In the following, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere in $\mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$, where e_1, \dots, e_n are the coordinate unit vectors. For $\theta \in S^{n-1}$, $d\theta$ denotes the surface element on S^{n-1} ; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of S^{n-1} . We set $d_*\theta = d\theta/\sigma_{n-1}$ for the normalized surface element on S^{n-1} .

The letter c denotes an inessential positive constant that may vary at each occurrence.

2. SOME PROPERTIES OF THE GEGENBAUER-CHEBYSHEV INTEGRALS

The Gegenbauer polynomials $C_m^\lambda(t)$, $\lambda > -1/2$, have the form

$$C_m^\lambda(t) = \sum_{j=0}^M c_{m,j} t^{m-2j}, \quad c_{m,j} = (-1)^j \frac{2^{m-2j} \Gamma(m-j+\lambda)}{\Gamma(\lambda) j! (m-2j)!}, \quad (2.1)$$

where $M = [m/2]$ is the integer part of $m/2$; see [8]. In the case $\lambda = 0$, they are usually substituted by the Chebyshev polynomials $T_m(t)$. If $|t| \leq 1$, then

$$|C_m^\lambda(t)| \leq c \begin{cases} 1, & \text{if } m \text{ is even,} \\ |t|, & \text{if } m \text{ is odd,} \end{cases} \quad c \equiv c(\lambda, m) = \text{const.} \quad (2.2)$$

The same inequality holds for $T_m(t)$; cf. 10.9(18) and 10.11(22) in [8].

2.1. The right-sided integrals. The *right-sided Gegenbauer-Chebyshev integrals* of a function f on \mathbb{R}_+ are defined by

$$(\mathcal{G}_-^{\lambda,m} f)(t) = \frac{1}{c_{\lambda,m}} \int_t^\infty (r^2 - t^2)^{\lambda-1/2} C_m^\lambda\left(\frac{t}{r}\right) f(r) r dr, \quad (2.3)$$

$$(\mathcal{G}_-^{*\lambda,m} f)(t) = \frac{t}{c_{\lambda,m}} \int_t^\infty (r^2 - t^2)^{\lambda-1/2} C_m^\lambda\left(\frac{r}{t}\right) f(r) \frac{dr}{r^{2\lambda+1}}, \quad (2.4)$$

$$c_{\lambda,m} = \frac{\Gamma(2\lambda + m) \Gamma(\lambda + 1/2)}{2m! \Gamma(2\lambda)}, \quad \lambda > -1/2, \quad \lambda \neq 0. \quad (2.5)$$

In the case $\lambda = 0$, when the Gegenbauer polynomials are substituted by the Chebyshev ones, we set

$$(\mathcal{T}_-^m f)(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty (r^2 - t^2)^{-1/2} T_m\left(\frac{t}{r}\right) f(r) r dr, \quad (2.6)$$

$$(\mathcal{T}_-^{*m} f)(t) = \frac{2t}{\sqrt{\pi}} \int_t^\infty (r^2 - t^2)^{-1/2} T_m\left(\frac{r}{t}\right) f(r) \frac{dr}{r}. \quad (2.7)$$

In the following, all statements are presented for the case of Gegenbauer polynomials $C_m^\lambda(t)$. The corresponding statements for the Chebyshev polynomials can be formally obtained by setting $\lambda = 0$ and proved similarly.

Proposition 2.1. [22, Proposition 3.1] *Let $a > 0$, $\lambda > -1/2$. The integrals $(\mathcal{G}_-^{\lambda,m} f)(t)$ and $(\mathcal{G}_-^{*\lambda,m} f)(t)$ are finite for almost all $t > a$ under the following conditions.*

(i) For $(\mathcal{G}_-^{\lambda,m} f)(t)$:

$$\int_a^\infty |f(t)| t^{2\lambda-\eta} dt < \infty, \quad \eta = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \quad (2.8)$$

(ii) For $(\mathcal{G}_-^{*\lambda,m} f)(t)$:

$$\int_a^\infty |f(t)| t^{m-2} dt < \infty. \quad (2.9)$$

Proposition 2.2. *If $f \in S(\mathbb{R}_+)$, then $\mathcal{G}_-^{\lambda,m} f$ is an infinitely differentiable function on \mathbb{R}_+ such that $t^{\gamma-1} \mathcal{G}_-^{\lambda,m} f \in L^1(\mathbb{R}_+)$ for all $\gamma > 0$.*

Proof. The infinite differentiability of $(\mathcal{G}_-^{\lambda,m} f)(t)$ is easily seen if we write

$$(\mathcal{G}_-^{\lambda,m} f)(t) = \frac{t^{2\lambda+1}}{c_{\lambda,m}} \int_1^\infty (s^2 - 1)^{\lambda-1/2} C_m^\lambda\left(\frac{1}{s}\right) f(ts) s ds.$$

The second statement can be checked straightforward by making use of (2.2). \square

Proposition 2.3. *The operator $\mathcal{G}_-^{*\lambda,m}$ is injective on the class of continuous functions f on \mathbb{R}_+ satisfying*

$$t^{\gamma-1} f(t) \in L^1(\mathbb{R}_+) \quad \text{for some } \gamma > m - 1. \quad (2.10)$$

Proof. We write $\mathcal{G}_-^{*\lambda,m} f$ as a Mellin convolution

$$(\mathcal{G}_-^{*\lambda,m} f)(t) = \int_0^\infty g(s) f\left(\frac{t}{s}\right) \frac{ds}{s}, \quad g(s) = \frac{s}{c_{\lambda,m}} (1-s^2)_+^{\lambda-1/2} C_m^\lambda\left(\frac{1}{s}\right).$$

By the formula 2.21.2(25) from [17],

$$\begin{aligned} \tilde{g}(z) &= \int_0^\infty g(s) s^{z-1} ds \\ &= \frac{\Gamma\left(\frac{z+1-m}{2}\right) \Gamma\left(\lambda + \frac{z+1+m}{2}\right)}{\Gamma\left(\lambda + \frac{z+1}{2}\right) \Gamma\left(\lambda + \frac{z+2}{2}\right)}, \quad \operatorname{Re} z > m-1. \end{aligned}$$

Hence, f is uniquely reconstructed by the Mellin inversion formula (see, e.g., [15])

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{f}(z) \tilde{g}(z) dz, \quad \gamma > m-1,$$

which gives the desired injectivity result. \square

We will also need the Riemann-Liouville fractional integrals

$$(I_-^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{f(s) ds}{(s-t)^{1-\alpha}}, \quad t > 0, \quad \alpha > 0. \quad (2.11)$$

The corresponding operators of fractional differentiation, which are defined as the left inverses of I_-^α , will be denoted by D_-^α .

Proposition 2.4. *The operator I_-^α is a bijection of $S(\mathbb{R}_+)$ onto itself.*

Proof. This fact is not new, and the proof is given for the sake of completeness. Let $f \in S(\mathbb{R}_+)$,

$$\varphi(t) = (I_-^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} f(s+t) ds.$$

This function is infinitely differentiable for $t > 0$ and all derivatives $\varphi^{(\ell)}(t) = (I_-^\alpha f^{(\ell)})(t)$ have finite limit as $t \rightarrow 0^+$. Thus, $I_-^\alpha f \in S(\mathbb{R}_+)$. The injectivity of I_-^α is a standard fact from Fractional Calculus; see, e.g., [20, 24], so that $D_-^\alpha I_-^\alpha f = f$. Here D_-^α has different forms, for instance, $(D_-^\alpha \varphi)(t) = (-d/dt)^m (I_-^{m-\alpha} \varphi)(t)$ for any integer $m > \alpha$.

Conversely, given any $\varphi \in S(\mathbb{R}_+)$ and $\alpha > 0$, for any integer $m > \alpha$ we have

$$\varphi(t) = - \int_t^\infty \varphi'(s) ds = \cdots = (-1)^m (I_-^m f^{(m)})(t) = (I_-^\alpha \psi)(t),$$

where $\psi = (-1)^m I_-^{m-\alpha} f^{(m)} \in S(\mathbb{R}_+)$ by the first part of the proof. Hence, $I_-^\alpha : S(\mathbb{R}_+) \rightarrow S(\mathbb{R}_+)$ is surjective. \square

Proposition 2.5. [22, Lemma 3.4] *Let $\lambda > -1/2$, $m \geq 2$. Suppose that*

$$\int_a^\infty |f(t)| t^{2\lambda+m-1} dt < \infty \quad \forall a > 0. \quad (2.12)$$

Then for almost all $t > 0$,

$$(\mathcal{G}_-^{*\lambda,m} \mathcal{G}_-^{\lambda,m} f)(t) = 2^{2\lambda+1} (I_-^{2\lambda+1} f)(t). \quad (2.13)$$

Definition 2.6. *We denote by $\mathcal{D}_m(\mathbb{R}_+)$ the set of all functions $\varphi \in C_c^\infty(\mathbb{R}_+)$ satisfying the moment conditions*

$$\int_0^\infty r^{m-2k} \varphi(r) dr = 0 \quad \forall 1 \leq k \leq M, \quad M = [m/2]. \quad (2.14)$$

Proposition 2.7. *If $\varphi \in \mathcal{D}_m(\mathbb{R}_+)$, then $\mathcal{G}_-^{*\lambda,m} \varphi \in S(\mathbb{R}_+)$. Moreover, if $\text{supp } \varphi \subset [a, b]$, $0 < a < b < \infty$, then $(\mathcal{G}_-^{*\lambda,m} \varphi)(t) = 0$ for all $t > b$.*

Proof. The second statement is an immediate consequence of the right-sided structure of $\mathcal{G}_-^{*\lambda,m} \varphi$. To prove the first statement, let $f = \mathcal{G}_-^{*\lambda,m} \varphi$. Then

$$f(t) = \frac{1}{c_{\lambda,m}} \int_0^1 (1-s^2)^{\lambda-1/2} C_m^\lambda \left(\frac{1}{s} \right) \varphi \left(\frac{t}{s} \right) ds.$$

Since φ is compactly supported, this function is infinitely differentiable for $t > 0$ and we can write

$$f^{(\ell)}(t) = \frac{1}{2 c_{\lambda,m}} \int_0^1 (1-x)^{\lambda-1/2} C_m^\lambda \left(\frac{1}{\sqrt{x}} \right) \varphi^{(\ell)} \left(\frac{t}{\sqrt{x}} \right) \frac{dx}{(\sqrt{x})^{\ell+1}}.$$

By Taylor's formula, $(1-x)^{\lambda-1/2} = p_n(x) + c x^{n+1} \omega_n(x)$, where $p_n(x)$ is a polynomial of degree n , $c = \text{const}$, and

$$\omega_n(x) = \int_0^1 (1-yx)^{\lambda-n-3/2} (1-y)^n dy.$$

Hence (cf. (2.1)), for some $N > 0$,

$$(1-x)^{\lambda-1/2} C_m^\lambda \left(\frac{1}{\sqrt{x}} \right) = \sum_{j=0}^N c_j (\sqrt{x})^{2j-m} + c x^{n+1} \omega_n(x) C_m^\lambda \left(\frac{1}{\sqrt{x}} \right).$$

Plugging this expression in $f^{(\ell)}$, we obtain $f^{(\ell)} = A + B$, where A is a linear combination of integrals

$$A_{j,\ell}(t) = \int_0^1 (\sqrt{x})^{2j-m-\ell-1} \varphi^{(\ell)} \left(\frac{t}{\sqrt{x}} \right) dx = c_{j,\ell} t^{2j-m-\ell+1},$$

$$c_{j,\ell} = 2 \int_t^\infty r^{m+\ell-2j-2} \varphi^{(\ell)}(r) dr, \quad j = 0, 1, \dots, N; \quad \ell = 0, 1, \dots,$$

and

$$\begin{aligned} B &= c \int_0^1 \omega_n(x) C_m^\lambda \left(\frac{1}{\sqrt{x}} \right) \varphi^{(\ell)} \left(\frac{t}{\sqrt{x}} \right) x^{n+1-(\ell+1)/2} dx \\ &= 2c t^{2n-\ell+3} \int_t^\infty \omega_n \left(\frac{t^2}{r^2} \right) C_m^\lambda \left(\frac{r}{t} \right) \varphi^{(\ell)}(r) r^{\ell-2n-4} dr. \end{aligned}$$

If φ satisfies (2.14), then all $A_{j,\ell}$ have a finite limit as $t \rightarrow 0$. Furthermore, by (2.1), B is a linear combination of the integrals

$$B_{j,\ell} = t^{2n-\ell+3+2j-m} \int_t^\infty \omega_n \left(\frac{t^2}{r^2} \right) \varphi^{(\ell)}(r) r^{\ell-2n-4+m-2j} dr.$$

Recall that $\text{supp } \varphi \subset [a, b]$, $0 < a < b < \infty$, and suppose that $0 < t < a/2$. Then

$$\begin{aligned} \omega_n \left(\frac{t^2}{r^2} \right) &= \int_0^1 \left(1 - \frac{yt^2}{r^2} \right)^{\lambda-n-3/2} (1-y)^n dy \\ &\leq \int_0^1 \frac{(1-y)^n dy}{(1-y/4)^{n+3/2-\lambda}} = c < \infty. \end{aligned}$$

It follows that $B_{j,\ell} = O(t^{2n-\ell+3+2j-m}) \rightarrow 0$ as $t \rightarrow 0$ if n is big enough, which completes the proof. \square

Proposition 2.8. *Every function $\varphi \in \mathcal{D}_m(\mathbb{R}_+)$ is represented as $\varphi = \mathcal{G}_-^{\lambda,m} \psi$, where*

$$\psi = 2^{-2\lambda-1} D_-^{2\lambda+1} \mathcal{G}_-^{*\lambda,m} \varphi \in S(\mathbb{R}_+). \quad (2.15)$$

If $\text{supp } \varphi \subset [a, b]$, $0 < a < b < \infty$, then $\varphi(t) = 0$ for all $t > b$.

Proof. By Proposition 2.7, the function $f = \mathcal{G}_-^{*\lambda,m} \varphi$ belongs to $S(\mathbb{R}_+)$ and equals zero for all $t > b$. Hence, by Proposition 2.4, the function $\psi = 2^{-2\lambda-1} D_-^{2\lambda+1} f$ also belongs to $S(\mathbb{R}_+)$ and equals zero for all $t > b$. To show that $\varphi = \mathcal{G}_-^{\lambda,m} \psi$, let $F = \varphi - \mathcal{G}_-^{\lambda,m} \psi$. By Proposition 2.2, $F \in C^\infty(\mathbb{R}_+)$ and $t^{\gamma-1} F \in L^1(\mathbb{R}_+)$ for all $\gamma > 0$. Hence, owing to Propositions 2.5 and 2.1,

$$\mathcal{G}_-^{*\lambda,m} F = \mathcal{G}_-^{*\lambda,m} \varphi - \mathcal{G}_-^{*\lambda,m} \mathcal{G}_-^{\lambda,m} \psi = \mathcal{G}_-^{*\lambda,m} \varphi - I_-^{2\lambda+1} D_-^{2\lambda+1} \mathcal{G}_-^{*\lambda,m} \varphi = 0.$$

Now, the required result follows from the injectivity of the operator $\mathcal{G}_-^{*\lambda,m}$; see Proposition 2.3. \square

2.2. The left-sided integrals. Let $\lambda > -1/2$, $m \in \mathbb{Z}_+$, $0 < a \leq \infty$. The *left-sided Gegenbauer and Chebyshev fractional integrals* on the interval $(0, a)$ are defined as follows. For $\lambda \neq 0$, we set

$$(\mathcal{G}_+^{\lambda,m} f)(r) = \frac{r^{-2\lambda}}{c_{\lambda,m}} \int_0^r (r^2 - t^2)^{\lambda-1/2} C_m^\lambda \left(\frac{t}{r} \right) f(t) dt, \quad (2.16)$$

$$(\mathcal{G}_+^{*\lambda,m} f)(r) = \frac{1}{c_{\lambda,m}} \int_0^r (r^2 - t^2)^{\lambda-1/2} C_m^\lambda \left(\frac{r}{t} \right) f(t) t dt, \quad (2.17)$$

$c_{\lambda,m}$ being defined by (2.5), $0 < r < a$. In the case $\lambda = 0$ we denote

$$(\mathcal{T}_+^m f)(r) = \frac{2}{\sqrt{\pi}} \int_0^r (r^2 - t^2)^{-1/2} T_m \left(\frac{t}{r} \right) f(t) dt, \quad (2.18)$$

$$(\mathcal{T}_+^{*m} f)(r) = \frac{2}{\sqrt{\pi}} \int_0^r (r^2 - t^2)^{-1/2} T_m \left(\frac{r}{t} \right) f(t) t dt. \quad (2.19)$$

The left-sided integrals are expressed through the right-sided ones by the formulas

$$(\mathcal{G}_+^{\lambda,m} f)(r) = \frac{1}{r} (\mathcal{G}_-^{\lambda,m} f_1) \left(\frac{1}{r} \right), \quad f_1(t) = \frac{1}{t^{2\lambda+2}} f \left(\frac{1}{t} \right); \quad (2.20)$$

$$(\mathcal{G}_+^{*\lambda,m} f)(r) = r^{2\lambda} (\mathcal{G}_-^{*\lambda,m} f_2) \left(\frac{1}{r} \right), \quad f_2(t) = \frac{1}{t} f \left(\frac{1}{t} \right). \quad (2.21)$$

These formulas combined with Proposition 2.1 give the following statement.

Proposition 2.9. *Let $a > 0$, $\lambda > -1/2$. The integrals (2.16)-(2.19) are absolutely convergent for almost all $r < a$ under the following conditions.*

(i) For (2.16), (2.18):

$$\int_0^a t^\eta |f(t)| dt < \infty, \quad \eta = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \quad (2.22)$$

(ii) For (2.17), (2.19):

$$\int_0^a t^{1-m} |f(t)| dt < \infty. \quad (2.23)$$

Lemma 2.10. [22, Proposition 3.7] *If $m = 0, 1$, then $\mathcal{G}_+^{\lambda,m}$ is injective on \mathbb{R}_+ in the class of functions satisfying (2.22) for all $a > 0$. If $m \geq 2$, then $\mathcal{G}_+^{\lambda,m}$ is non-injective in this class of functions. Specifically, let $f_k(t) = t^k$, where k is a nonnegative integer such that $m - k = 2, 4, \dots$. Then $(\mathcal{G}_+^{\lambda,m} f_k)(t) = 0$ for all $t > 0$.*

An important question is: Are there any other functions in the kernel of the operator $\mathcal{G}_+^{\lambda,m}$ rather than f_k in Lemma 2.10? Below we give a negative answer to this question under certain conditions, which are very close to (2.22). Let

$$\varkappa_{\lambda,m}(t) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ t^{1+2\lambda} & \text{if } m \text{ is odd, } -1/2 < \lambda < 0, \\ t(1 + |\log t|) & \text{if } m \text{ is odd, } \lambda = 0, \\ t & \text{if } m \text{ is odd, } \lambda > 0. \end{cases} \quad (2.24)$$

Given $0 < a \leq \infty$, we denote by $L_\varkappa^1(0, a)$ the set of all function f on $(0, a)$ such that

$$\int_0^{a_1} |f(t)| \varkappa_{\lambda,m}(t) dt \leq \infty \quad \forall a_1 < a. \quad (2.25)$$

Clearly, $L_{loc}^1(0, a) \subset L_{\varkappa}^1(0, a)$.

Lemma 2.11. *Suppose that $m \geq 2$, $M = [m/2]$, $0 < a \leq \infty$. If $f \in L_{\varkappa}^1(0, a)$ and $(\mathcal{G}_+^{\lambda, m} f)(r) = 0$ for almost all $0 < r < a$, then*

$$f(t) = \sum_{j=1}^M c_j t^{m-2j} \quad \text{a.e. on } (0, a) \quad (2.26)$$

with some coefficients c_j .

Proof. We introduce auxiliary functions

$$\varphi_{m-2k} \in C_c^\infty(0, a), \quad k \in \{1, 2, \dots, M\},$$

satisfying

$$\int_0^a t^{m-2j} \varphi_{m-2k}(t) dt = \delta_{j,k} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.27)$$

The existence of such functions is a consequence of the general fact from functional analysis for bi-orthogonal systems; see, e.g., [12, p. 160]. We set

$$c_j = \int_0^a f(t) \varphi_{m-2j}(t) dt. \quad (2.28)$$

Let $\omega \in C_c^\infty(0, a)$ be an arbitrary test function, and let

$$\varphi(t) = \omega(t) - \sum_{j=1}^M \varphi_{m-2j}(t) \int_0^a s^{m-2j} \omega(s) ds \quad (\in C_c^\infty(0, a)). \quad (2.29)$$

Then for any $k \in \{1, 2, \dots, M\}$,

$$\begin{aligned} \int_0^a t^{m-2k} \varphi(t) dt &= \int_0^a t^{m-2k} \omega(t) dt \\ &\quad - \sum_{j=1}^M \left[\int_0^a t^{m-2k} \varphi_{m-2j}(t) dt \right] \left[\int_0^a s^{m-2j} \omega(s) ds \right]. \end{aligned}$$

By (2.27), this gives

$$\int_0^a t^{m-2k} \varphi(t) dt = 0 \quad \forall k \in \{1, 2, \dots, M\}.$$

Suppose that $\varphi(t) \equiv 0$ on some interval (a_1, a) with $a_1 < a$, and define $\tilde{\varphi} \in C_c^\infty(\mathbb{R}_+)$ so that $\tilde{\varphi}(t) = \varphi(t)$ if $t \leq a_1$ and $\tilde{\varphi}(t) = 0$ if $t > a_1$.

By Proposition 2.8, $\tilde{\varphi}$ is represented as $\tilde{\varphi} = \mathcal{G}_-^{\lambda, m} \psi$, where ψ belongs to $S(\mathbb{R}_+)$ and equals zero on (a_1, ∞) . Then

$$\begin{aligned} \int_0^\infty f(t) \tilde{\varphi}(t) dt &= \int_0^\infty f(t) (\mathcal{G}_-^{\lambda, m} \psi)(t) dt \\ &= \int_0^{a_1} \psi(r) (\mathcal{G}_+^{\lambda, m} f)(r) r^{2\lambda+1} dr = 0. \end{aligned} \quad (2.30)$$

Hence, by (2.29), (2.30), and (2.28),

$$\begin{aligned} \int_0^a f(t) \omega(t) dt &= \int_0^\infty f(t) \left[\varphi(t) + \sum_{j=1}^M \varphi_{m-2j}(t) \int_0^a s^{m-2j} \omega(s) ds \right] dt \\ &= \sum_{j=1}^M c_j \int_0^a s^{m-2j} \omega(s) ds = \int_0^a \left[\sum_{j=1}^M c_j s^{m-2j} \right] \omega(s) ds. \end{aligned}$$

This gives (2.26).

To complete the proof, we must justify application of Fubini's theorem in (2.30). Replacing $(\mathcal{G}_+^{\lambda, m} f)(r)$ according to (2.16), and taking absolute values, we have

$$\begin{aligned} I &\equiv \int_0^{a_1} |\psi(r)| r dr \int_0^r (r^2 - t^2)^{\lambda-1/2} \left| C_m^\lambda \left(\frac{t}{r} \right) \right| |f(t)| dt \\ &\leq c \int_0^{a_1} r dr \int_0^r (r^2 - t^2)^{\lambda-1/2} \left(\frac{t}{r} \right)^\eta |f(t)| dt, \end{aligned}$$

where $\eta = 0$ if m is even and $\eta = 1$ if m is odd. If $\eta = 0$, then

$$I \leq c \int_0^{a_1} |f(t)| dt \int_t^{a_1} (r^2 - t^2)^{\lambda-1/2} r dr \leq c_1 \int_0^{a_1} |f(t)| dt < \infty.$$

If $\eta = 1$, then

$$I \leq c \int_0^{a_1} |f(t)| t g(t) dt,$$

where

$$g(t) = \int_t^{a_1} (r^2 - t^2)^{\lambda-1/2} dr = t^{2\lambda} \int_1^{a_1/t} (s^2 - 1)^{\lambda-1/2} ds.$$

The behavior of $g(t)$ as $t \rightarrow 0$ can be easily examined by considering the cases indicated in (2.24). This gives

$$I \leq c \int_0^{a_1} |f(t)| \kappa_{\lambda,m}(t) dt < \infty.$$

□

Lemma 2.11 combined with (2.20) gives the following result for the right-sided Gegenbauer-Chebyshev integrals.

Let

$$\tilde{\kappa}_{\lambda,m}(t) = \begin{cases} t^{2\lambda} & \text{if } m \text{ is even,} \\ t^{-1} & \text{if } m \text{ is odd, } -1/2 < \lambda < 0, \\ t^{2\lambda-1} (1 + |\log t|) & \text{if } m \text{ is odd, } \lambda = 0, \\ t^{2\lambda-1} & \text{if } m \text{ is odd, } \lambda > 0. \end{cases} \quad (2.31)$$

Given $a \geq \infty$, we denote by $L_{\tilde{\kappa}}^1(a, \infty)$ the set of all function f on (a, ∞) such that

$$\int_{a_1}^{\infty} |f(t)| \tilde{\kappa}_{\lambda,m}(t) dt \leq \infty \quad \forall a_1 > a. \quad (2.32)$$

Lemma 2.12. *Suppose that $m \geq 2$, $M = [m/2]$, $a \geq 0$. If $f \in L_{\tilde{\kappa}}^1(a, \infty)$ and $(\mathcal{G}_{-}^{\lambda,m} f)(r) = 0$ for almost all $r > a$, then*

$$f(t) = \sum_{k=0}^{M-1} c_k t^{2k-m-2\lambda} \quad \text{a.e. on } (a, \infty) \quad (2.33)$$

with some coefficients c_k .

Lemma 2.12 is an important complement of the following statement, which was proved in [22, Lemma 3.3].

Lemma 2.13. *Let $\lambda > -1/2$. If $m = 0, 1$, then $\mathcal{G}_{-}^{\lambda,m}$ is injective on \mathbb{R}_{+} in the class of functions satisfying (2.8) for all $a > 0$. If $m \geq 2$, then $\mathcal{G}_{-}^{\lambda,m}$ is non-injective in this class of functions. Specifically, let $f_k(t) = t^{-2\lambda-k-2}$, where k is a nonnegative integer such that $m - k = 2, 4, \dots$. Then $(\mathcal{G}_{-}^{\lambda,m} f_k)(t) = 0$ for all $t > 0$.*

According to Lemma 2.12, the functions f_k exhaust the kernel of the operator $\mathcal{G}_{-}^{\lambda,m}$ in the space $L_{\tilde{\kappa}}^1(a, \infty)$, $a \geq 0$.

3. RADON TRANSFORMS ON \mathbb{R}^n

We recall some known facts; see, e.g., [9, 11, 21, 23]. Let Π_n be the set of all unoriented hyperplanes in \mathbb{R}^n . The Radon transform of a function f on \mathbb{R}^n is defined by the formula

$$(Rf)(\xi) = \int_{\xi} f(x) d_{\xi}x, \quad \xi \in \Pi_n, \quad (3.1)$$

provided that this integral exists. Here $d_{\xi}x$ denotes the Euclidean volume element in ξ . Every hyperplane $\xi \in \Pi_n$ has the form $\xi = \{x : x \cdot \theta = t\}$, where $\theta \in S^{n-1}$, $t \in \mathbb{R}$. Thus, we can write (3.1) as

$$(Rf)(\theta, t) = \int_{\theta^{\perp}} f(t\theta + u) d_{\theta}u, \quad (3.2)$$

where $\theta^{\perp} = \{x : x \cdot \theta = 0\}$ is the hyperplane orthogonal to θ and passing through the origin, $d_{\theta}u$ is the Euclidean volume element in θ^{\perp} . We set $Z_n = S^{n-1} \times \mathbb{R}$ and equip Z_n with the product measure $d_*\theta dt$, where $d_*\theta = \sigma_{n-1}^{-1}d\theta$ is the normalized surface measure on S^{n-1} .

Clearly, $(Rf)(\theta, t) = (Rf)(-\theta, -t)$ for every $(\theta, t) \in Z_n$.

Theorem 3.1. (cf. [21, Theorem 3.2]) *If*

$$\int_{|x|>a} \frac{|f(x)|}{|x|} dx < \infty \quad \forall a > 0, \quad (3.3)$$

then $(Rf)(\xi)$ is finite for almost all $\xi \in \Pi_n$. If f is nonnegative, radial, and (3.3) fails, then $(Rf)(\xi) \equiv \infty$.

The dual Radon transform is an averaging operator that takes a function $\varphi(\theta, t)$ on Z_n to a function $(R^*\varphi)(x)$ on \mathbb{R}^n by the formula

$$(R^*\varphi)(x) = \int_{S^{n-1}} \varphi(\theta, x \cdot \theta) d_*\theta. \quad (3.4)$$

The operators R and R^* can be expressed one through another.

Lemma 3.2. [22, Lemma 2.6] *Let $x \neq 0$, $t \neq 0$,*

$$(A\varphi)(x) = \frac{1}{|x|^n} \varphi\left(\frac{x}{|x|}, \frac{1}{|x|}\right), \quad (Bf)(\theta, t) = \frac{1}{|t|^n} f\left(\frac{\theta}{t}\right). \quad (3.5)$$

The following equalities hold provided that the expressions on either side exist in the Lebesgue sense:

$$(R^*\varphi)(x) = \frac{2}{|x| \sigma_{n-1}} (RA\varphi)\left(\frac{x}{|x|}, \frac{1}{|x|}\right), \quad (3.6)$$

$$(Rf)(\theta, t) = \frac{\sigma_{n-1}}{2|t|} (R^* Bf) \left(\frac{\theta}{t} \right). \quad (3.7)$$

Theorem 3.1 combined with (3.6) gives the following

Corollary 3.3. *If $\varphi(\theta, t)$ is locally integrable on Z_n , then the dual Radon transform $(R^*\varphi)(x)$ is finite for almost all $x \in \mathbb{R}^n$. If $\varphi(\theta, t)$ is nonnegative, independent of θ , i.e., $\varphi(\theta, t) \equiv \varphi_0(t)$, and such that*

$$\int_0^a \varphi_0(t) dt = \infty,$$

for some $a > 0$, then $(R^*\varphi)(x) \equiv \infty$.

We fix a real-valued orthonormal basis $\{Y_{m,\mu}\}$ of spherical harmonics in $L^2(S^{n-1})$; see, e.g., [16]. Here $m \in \mathbb{Z}_+$ and $\mu = 1, 2, \dots, d_n(m)$, where

$$d_n(m) = (n + 2m - 2) \frac{(n + m - 3)!}{m! (n - 2)!} \quad (3.8)$$

is the dimension of the subspace of spherical harmonics of degree m .

Lemma 3.4. [22, Lemma 4.3] *Let $\lambda = (n - 2)/2$, $\varphi(\theta, t) = v(t) Y_m(\theta)$, where Y_m is a spherical harmonic of degree m and $v(t)$ is a locally integrable function on \mathbb{R} satisfying $v(-t) = (-1)^m v(t)$. Then $(R^*\varphi)(x) \equiv (R^*\varphi)(r\theta)$ is finite for all $\theta \in S^{n-1}$ and almost all $r > 0$. Furthermore,*

$$(R^*\varphi)(r\theta) = u(r) Y_m(\theta). \quad (3.9)$$

The function $u(r)$ is represented by the Gegenbauer integral (2.16) (or the Chebyshev integral (2.18)) as follows.

For $n \geq 3$:

$$u(r) = \frac{r^{-2\lambda}}{\tilde{c}_{\lambda,m}} \int_0^r (r^2 - t^2)^{\lambda-1/2} C_m^\lambda \left(\frac{t}{r} \right) v(t) dt = \pi^{\lambda+1/2} (\mathcal{G}_+^{\lambda,m} v)(t), \quad (3.10)$$

$$\tilde{c}_{\lambda,m} = \frac{\pi^{1/2} \Gamma(2\lambda + m) \Gamma(\lambda + 1/2)}{2m! \Gamma(2\lambda) \Gamma(\lambda + 1)}.$$

For $n = 2$:

$$u(r) = \frac{2}{\pi} \int_0^r (r^2 - t^2)^{-1/2} T_m \left(\frac{t}{r} \right) v(t) dt = \pi^{1/2} (\mathcal{T}_+^m v)(t). \quad (3.11)$$

In parallel with the Radon transforms Rf and $R^*\varphi$ defined above, we shall also consider their exterior and interior versions, respectively. For $a > 0$, we denote

$$B_a^+ = \{x \in \mathbb{R}^n : |x| < a\}, \quad B_a^- = \{x \in \mathbb{R}^n : |x| > a\},$$

$$C_a^+ = \{(\theta, t) \in Z_n : |t| < a\}, \quad C_a^- = \{(\theta, t) \in Z_n : |t| > a\}.$$

When dealing with the *exterior Radon transform*, we assume that f is defined on B_a^- and $(Rf)(\theta, t)$ is considered for $(\theta, t) \in C_a^-$. Similarly, in the study of the *interior dual Radon transform*, it is supposed that φ is defined on C_a^+ and the values of $(R^*\varphi)(x)$ lie in B_a^+ .

Lemma 3.4 implies the following statement; cf. [22, Lemma 4.7].

Lemma 3.5. *Let $0 < a \leq \infty$. If $\varphi \in L_{loc}^1(C_a^+)$ is even, then for almost all $r \in (0, a)$,*

$$(R^*\varphi)_{m,\mu}(r) \equiv \int_{S^{n-1}} (R^*\varphi)(r\theta) Y_{m,\mu}(\theta) d\theta = \pi^{\lambda+1/2} (\mathcal{G}_+^{\lambda,m} \varphi_{m,\mu})(r), \quad (3.12)$$

where $\lambda = (n-2)/2$ and $\mathcal{G}_+^{\lambda,m} \varphi_{m,\mu}$ is the Gegenbauer integral (2.16) (or the Chebyshev integral (2.18)).

The next theorem gives the description of the kernel of R^* in terms of the Fourier-Laplace coefficients

$$\varphi_{m,\mu}(t) = \int_{S^{n-1}} \varphi(\theta, t) Y_{m,\mu}(\theta) d\theta. \quad (3.13)$$

Theorem 3.6. *Let $\varphi(\theta, t)$ be an even locally integrable function on C_a^+ , $0 < a \leq \infty$.*

(i) *Suppose that $\varphi_{m,\mu}(t) = 0$ for almost all $t \in (0, a)$ if $m = 0, 1$, and $\varphi_{m,\mu}$ is a linear combination of the form*

$$\varphi_{m,\mu}(t) = \sum_{j=1}^M c_j t^{m-2j}, \quad c_j = \text{const}, \quad M = [m/2], \quad (3.14)$$

if $m \geq 2$. Then $(R^\varphi)(x) = 0$ for almost all $x \in B_a^+$.*

(ii) *Conversely, if $(R^*\varphi)(x) = 0$ for almost all $x \in B_a^+$, then, for all $\mu = 1, 2, \dots, d_n(m)$ and almost all $t < a$, the following statements hold.*

(a) *If $m = 0, 1$, then $\varphi_{m,\mu}(t) = 0$.*

(b) *If $m \geq 2$, then $\varphi_{m,\mu}(t)$ has the form (3.14) with some constants c_j .*

Proof. The statement (i) was proved in [22, Theorem 4.5]. To prove (ii), we observe that if $(R^*\varphi)(x) = 0$ for almost all $x \in B_a^+$, then, by (3.12), $(\mathcal{G}_+^{\lambda,m} \varphi_{m,\mu})(r) = 0$ for almost all $r \in (0, a)$ and all m, μ . Because $\varphi \in L_{loc}^1(C_a^+)$, then $\varphi_{m,\mu} \in L_{loc}^1(0, a)$, and (3.14) is an immediate consequence of Lemmas 2.10 and 2.11. \square

Theorem 3.6 together with Lemma 3.2 imply the following result for the Radon transform Rf .

Theorem 3.7. *Given $0 \leq a < \infty$, suppose that*

$$\int_{|x|>a_1} \frac{|f(x)|}{|x|} dx < \infty \quad \text{for all } a_1 > a, \quad (3.15)$$

and let $(Rf)(\theta, t) = 0$ almost everywhere on C_a^- . Then, for almost all $r > a$, all the Fourier-Laplace coefficients

$$f_{m,\mu}(r) = \int_{S^{n-1}} f(r\theta) Y_{m,\mu}(\theta) d\theta$$

have the form

$$f_{m,\mu}(r) = \begin{cases} 0 & \text{if } m=0, 1, \\ \sum_{j=1}^{[m/2]} c_j r^{2j-m-n} & \text{if } m \geq 2, \end{cases} \quad (3.16)$$

with some constants c_j . Conversely, for any constants c_j and any f satisfying (3.15) and (3.16), we have $(Rf)(\theta, t) = 0$ almost everywhere on C_a^- .

Proof. Suppose $a > 0$ (if $a = 0$ the changes are obvious). Note that (3.15) implies $(Bf)(\theta, t) \equiv |t|^{-n} f(\theta/t) \in L_{loc}^1(C_{1/a}^+)$. If $Rf = 0$ a.e. on C_a^- , then $R^*Bf = 0$ a.e. on $B_{1/a}^+$. Hence, by Theorem 3.6, for $t \in (0, 1/a)$ we have

$$(Bf)_{m,\mu}(t) = t^{-n} \int_{S^{n-1}} f\left(\frac{\theta}{t}\right) Y_{m,\mu}(\theta) d\theta = \begin{cases} 0 & \text{if } m = 0, 1, \\ \sum_{j=1}^M c_j t^{m-2j} & \text{if } m \geq 2. \end{cases}$$

Changing variable $t = 1/r$, we obtain (3.16). Conversely, if $f_{m,\mu}(r) = 0$ for $m = 0, 1$, and (3.16) holds for $m \geq 2$, then $(Bf)_{m,\mu}(t) = 0$ if $m = 0, 1$, and $(Bf)_{m,\mu}(t) = \sum_{j=1}^M c_j t^{m-2j}$ if $m \geq 2$. The last equality is obvious for $t > 0$. If $t < 0$, then

$$\begin{aligned} (Bf)_{m,\mu}(t) &= \int_{S^{n-1}} (Bf)(\theta, t) Y_{m,\mu}(\theta) d\theta \\ &= (-1)^m \int_{S^{n-1}} (Bf)(\theta, |t|) Y_{m,\mu}(\theta) d\theta \\ &= (-1)^m \sum_{j=1}^M c_j |t|^{m-2j} = \sum_{j=1}^M c_j t^{m-2j}. \end{aligned}$$

Hence, by Theorem 3.6, $R^*Bf = 0$ a.e. on $B_{1/a}^+$ and therefore, by (3.7), $Rf = 0$ a.e. on C_a^- . \square

Remark 3.8. An interesting open problem is the following: *What would be the structure of the kernel of the Radon transform R if the action of R is considered on functions, which may not satisfy (3.15) ?*

This question is intimately connected with the remarkable result of Armitage and Goldstein [2] who proved that there is a nonconstant harmonic function h on \mathbb{R}^n , $n \geq 2$, such that $\int_\xi |h| < \infty$ and $\int_\xi h = 0$ for every hyperplane ξ ; see also Zalcman [25] ($n = 2$) and Armitage [1]. One can easily show that h does not satisfy (3.15). Indeed, suppose the contrary, and let $\{h_{m,\mu}(r)\}$ be the set of all Fourier-Laplace coefficients of $h(x) \equiv h(r\theta)$. Then for any $a > 0$ and $m \geq 2$,

$$\begin{aligned} \int_0^a |h_{m,\mu}(r)| dr &\leq \int_0^a dr \int_{S^{n-1}} |h(r\theta)Y_{m,\mu}(\theta)| d\theta \\ &= \int_{|x|<a} |h(x)Y_{m,\mu}(x/|x|)| \frac{dx}{|x|^{n-1}} \leq c \int_{|x|<a} \frac{dx}{|x|^{n-1}} < \infty. \end{aligned}$$

On the other hand, the inequality

$$\int_0^a \left| \sum_{j=1}^M c_j r^{2j-m-n} \right| dr < \infty$$

is possible only if all c_j are zeros. The latter means that h consists only of harmonics of degree 0 and 1. However, by Theorem 3.7, these harmonics must be zero. Hence, $h(x) \equiv 0$, which contradicts the Armitage-Goldstein's result.

4. THE CORMACK-QUINTO TRANSFORM

The Cormack-Quinto transform

$$(\mathcal{Q}f)(x) = \int_{S^{n-1}} f(x + |x|\theta) d_*\theta \quad (4.1)$$

assigns to a function f on \mathbb{R}^n the mean values of f over spheres passing to the origin. In (4.1), f is integrated over the sphere of radius $|x|$ with center at x .

There is a remarkable connection between (4.1) and the dual Radon transform (3.4).

Lemma 4.1. [7, 22] *Let $n \geq 2$. Then*

$$(\mathcal{Q}f)(x) = |x|^{2-n}(R^*\varphi)(x), \quad \varphi(\theta, t) = (2|t|)^{n-2}f(2t\theta), \quad (4.2)$$

provided that either side of this equality exists in the Lebesgue sense.

A consequence of Lemma 4.1 is the following description of the kernel of \mathcal{Q} inherited from Theorem 3.6. We recall that the Fourier-Laplace coefficients of f are defined by

$$f_{m,\mu}(r) = \int_{S^{n-1}} f(r\theta) Y_{m,\mu}(\theta) d\theta, \quad r > 0. \quad (4.3)$$

Theorem 4.2. *Given $0 < a \leq \infty$, suppose that*

$$\int_{|x| < a_1} \frac{|f(x)|}{|x|} dx < \infty \quad \forall a_1 < 2a, \quad (4.4)$$

and let $(\mathcal{Q}f)(x) = 0$ for almost all $x \in B_a^+$. Then all the Fourier-Laplace coefficients (4.3) have the form

$$f_{m,\mu}(r) = \begin{cases} 0 & \text{if } m=0, 1, \\ \sum_{j=1}^{[m/2]} c_j r^{m-2j} & \text{if } m \geq 2, \end{cases} \quad (4.5)$$

for almost all $r \in (0, 2a)$ with some constants c_j . Conversely, for any constants c_j and any f satisfying (4.4) and (4.5), we have $(\mathcal{Q}f)(x) = 0$ a.e. on B_a^+ .

5. THE FUNK TRANSFORM

The Funk transform F assigns to a function f on a sphere the integrals of f over “equators”. Specifically, for the unit sphere S^n in \mathbb{R}^{n+1} we have

$$(Ff)(\theta) = \int_{\{\sigma \in S^n : \theta \cdot \sigma = 0\}} f(\sigma) d_\theta \sigma, \quad (5.1)$$

where $d_\theta \sigma$ stands for the $O(n+1)$ -invariant probability measure on the $(n-1)$ -dimensional section $\{\sigma \in S^n : \theta \cdot \sigma = 0\}$; see, e.g., [9, 11, 23].

Let e_1, \dots, e_{n+1} be the coordinate unit vectors, $\mathbb{R}^n = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_n$,

$$S_+^n = \{\theta = (\theta_1, \dots, \theta_{n+1}) \in S^n : 0 < \theta_{n+1} \leq 1\}. \quad (5.2)$$

It is known that $\ker F = \{0\}$ if the action of F is considered on even integrable functions. Below we prove that the structure of the kernel of F is different if the functions under consideration allow non-integrable singularities at the poles $\pm e_{n+1}$, so that the Funk transform still exists in the a.e. sense.

Consider the projection map

$$\mathbb{R}^n \ni x \xrightarrow{\mu} \theta \in S_+^n, \quad \theta = \mu(x) = \frac{x + e_{n+1}}{|x + e_{n+1}|}. \quad (5.3)$$

This map extends to the bijection $\tilde{\mu}$ from the set Π_n of all unoriented hyperplanes in \mathbb{R}^n onto the set

$$\tilde{S}_+^n = \{\theta = (\theta_1, \dots, \theta_{n+1}) \in S^n : 0 \leq \theta_{n+1} < 1\}. \quad (5.4)$$

cf. (5.2). Specifically, if $\tau = \{x \in \mathbb{R}^n : x \cdot \eta = t\} \in \Pi_n$, $\eta \in S^{n-1} \subset \mathbb{R}^n$, $t \geq 0$, and $\tilde{\tau}$ is the n -dimensional subspace containing the lifted plane $\tau + e_{n+1}$, then $\theta \in \tilde{S}_+^n$ is defined to be a normal vector to $\tilde{\tau}$.

The above notation is used in the following theorem.

Theorem 5.1. [22, Theorem 6.1] *Let $g(x) = (1 + |x|^2)^{-n/2} f(\mu(x))$, $x \in \mathbb{R}^n$, where f is an even function on S^n . The Funk transform F and the Radon transform R are related by the formula*

$$(Ff)(\theta) = \frac{2}{\sigma_{n-1} \sin d(\theta, e_{n+1})} (Rg)(\tilde{\mu}^{-1}\theta), \quad \theta \in \tilde{S}_+^n, \quad (5.5)$$

where $d(\theta, e_{n+1})$ is the geodesic distance between θ and e_{n+1} .

Theorems 3.7 and 5.1 yield the corresponding result for the kernel of the operator F . For f even, it suffices to consider the points $\theta \in S^n$, which are represented in the spherical polar coordinates as

$$\theta = \eta \sin \psi + e_{n+1} \cos \psi, \quad \eta \in S^{n-1}, \quad 0 < \psi < \pi/2.$$

The corresponding Fourier-Laplace coefficients (in the η -variable) have the form

$$f_{m,\mu}(\psi) = \int_{S^{n-1}} f(\eta \sin \psi + e_{n+1} \cos \psi) Y_{m,\mu}(\eta) d\eta. \quad (5.6)$$

Theorem 5.2. *Suppose that*

$$\int_{|\theta_{n+1}| < \alpha} |f(\theta)| d\theta < \infty \quad \text{for all } \alpha \in (0, 1) \quad (5.7)$$

and $(Ff)(\theta) = 0$ a.e. on S^n . Then all the Fourier-Laplace coefficients (5.6) have the form

$$f_{m,\mu}(\psi) \stackrel{\text{a.e.}}{=} \begin{cases} 0 & \text{if } m = 0, 1, \\ \sin^{-n} \psi \sum_{j=1}^{[m/2]} c_j \cot^{m-2j} \psi & \text{if } m \geq 2, \end{cases} \quad (5.8)$$

with some constants c_j . Conversely, for any constants c_j and any f satisfying (5.7) and (5.8), we have $(Ff)(\theta) = 0$ a.e. on S^n .

6. THE SPHERICAL SLICE TRANSFORM

The spherical slice transform

$$(\mathfrak{S}f)(\gamma) = \int_{\gamma} f(\eta) d_{\gamma}\eta, \quad (6.1)$$

assigns to a function f on S^n the integrals of f over $(n-1)$ -dimensional geodesic spheres $\gamma \subset S^n$ passing through the north pole e_{n+1} . Every geodesic sphere γ can be indexed by its center $\xi = (\xi_1, \dots, \xi_{n+1})$ in the closed hemisphere

$$\bar{S}_+^n = \{\xi = (\xi_1, \dots, \xi_{n+1}) \in S^n : 0 \leq \xi_{n+1} \leq 1\},$$

so that

$$\gamma \equiv \gamma(\xi) = \{\eta \in S^n : \eta \cdot \xi = e_{n+1} \cdot \xi\}, \quad \xi \in \bar{S}_+^n.$$

Using spherical polar coordinates, for $\xi \in \bar{S}_+^n$ we write

$$\xi = \theta \sin \psi + e_{n+1} \cos \psi, \quad \theta \in S^{n-1} \subset \mathbb{R}^n, \quad 0 \leq \psi \leq \pi/2,$$

$$\gamma \equiv \gamma(\xi) \equiv \gamma(\theta, \psi), \quad (\mathfrak{S}f)(\gamma) \equiv (\mathfrak{S}f)(\xi) \equiv (\mathfrak{S}f)(\theta, \psi).$$

Then

$$\gamma(\xi) = \{\eta \in S^n : \eta \cdot \xi = \cos \psi\}.$$

Consider the bijective mapping

$$\mathbb{R}^n \ni x \xrightarrow{\nu} \eta \in S^n \setminus \{e_{n+1}\}, \quad \nu(x) = \frac{2x + (|x|^2 - 1)e_{n+1}}{|x|^2 + 1}. \quad (6.2)$$

The inverse mapping $\nu^{-1} : S^n \setminus \{e_{n+1}\} \rightarrow \mathbb{R}^n$ is the stereographic projection from the north pole e_{n+1} onto $\mathbb{R}^n = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_n$. If

$$\eta = \omega \sin \varphi + e_{n+1} \cos \varphi, \quad \omega \in S^{n-1}, \quad 0 < \varphi \leq \pi,$$

then $x = \nu^{-1}(\eta) = s\omega$, $s = \cot(\varphi/2)$.

Lemma 6.1. [22, Lemma 7.2] *The spherical slice transform on S^n and the hyperplane Radon transform on \mathbb{R}^n are linked by the formula*

$$(\mathfrak{S}f)(\theta, \psi) = (Rg)(\theta, t), \quad t = \cot \psi, \quad (6.3)$$

$$g(x) = \left(\frac{2}{|x|^2 + 1} \right)^{n-1} (f \circ \nu)(x), \quad (6.4)$$

provided that either side of (6.3) is finite when f is replaced by $|f|$.

To describe the kernel of the operator \mathfrak{S} , we use the spherical harmonic decomposition of $f(\eta) = f(\omega \sin \varphi + e_{n+1} \cos \varphi)$ in the ω -variable. Let

$$f_{m,\mu}(\varphi) = \int_{S^{n-1}} f(\omega \sin \varphi + e_{n+1} \cos \varphi) Y_{m,\mu}(\omega) d\omega. \quad (6.5)$$

Then Theorems 3.7 and Lemma 6.1 imply the following statement.

Theorem 6.2. *Suppose that*

$$\int_{\eta_{n+1} > 1-\varepsilon} \frac{|f(\eta)|}{(1-\eta_{n+1})^{1/2}} d\eta < \infty \quad \forall \varepsilon \in (0, 2], \quad (6.6)$$

and $(\mathfrak{S}f)(\xi) = 0$ for almost all $\xi \in \bar{S}_+^n$. Then all the Fourier-Laplace coefficients (6.5) have the form

$$f_{m,\mu}(\varphi) \stackrel{\text{a.e.}}{=} \begin{cases} 0 & \text{if } m=0, 1, \\ (1-\cos \varphi)^{1-n} \sum_{j=1}^{[m/2]} c_j \left(\tan \frac{\varphi}{2}\right)^{n+m-2j} & \text{if } m \geq 2, \end{cases} \quad (6.7)$$

with some constants c_j . Conversely, for any constants c_j and any f satisfying (6.6) and (6.7), we have $(\mathfrak{S}f)(\xi) = 0$ a.e. on \bar{S}_+^n .

7. THE TOTALLY GEODESIC RADON TRANSFORM ON THE HYPERBOLIC SPACE

Let $\mathbb{E}^{n,1} \sim \mathbb{R}^{n+1}$, $n \geq 2$, be the $(n+1)$ -dimensional pseudo-Euclidean real vector space with the inner product

$$[x, y] = -x_1 y_1 - \dots - x_n y_n + x_{n+1} y_{n+1}. \quad (7.1)$$

The real hyperbolic space \mathbb{H}^n is realized as the upper sheet of the two-sheeted hyperboloid in $\mathbb{E}^{n,1}$, that is,

$$\mathbb{H}^n = \{x \in \mathbb{E}^{n,1} : [x, x] = 1, x_{n+1} > 0\};$$

see [10]. The corresponding one-sheeted hyperboloid is defined by

$$\mathbb{H}^{*n} = \{x \in \mathbb{E}^{n,1} : [x, x] = -1\}.$$

The totally geodesic Radon transform of a function f on \mathbb{H}^n is an integral operator of the form

$$(\mathfrak{R}f)(\xi) = \int_{\{x \in \mathbb{H}^n : [x, \xi] = 0\}} f(x) d_\xi x, \quad \xi \in \mathbb{H}^{*n}, \quad (7.2)$$

and represents an even function on \mathbb{H}^{*n} ; see [4, 10, 11].

Using the projective equivalence of the operator (7.2) and the hyperplane Radon transform R , as in [22, Lemma 8.3] (see also [13, 3]), we obtain the following kernel description, which is implied by Theorem 3.7. We write $x \in \mathbb{H}^n$ in the hyperbolic polar coordinates as $x = \theta \sinh r + e_{n+1} \cosh r$, $\theta \in S^{n-1}$, $r > 0$, and consider the Fourier-Laplace coefficients

$$f_{m,\mu}(r) = \int_{S^{n-1}} f(\theta \sinh r + e_{n+1} \cosh r) Y_{m,\mu}(\theta) d\theta. \quad (7.3)$$

Theorem 7.1. *Let*

$$\int_{x_{n+1} > 1+\delta} |f(x)| \frac{dx}{x_{n+1}} < \infty \quad \forall \delta > 0, \quad (7.4)$$

and let $(\mathfrak{R}f)(\xi) = 0$ a.e. on \mathbb{H}^n . Then all the Fourier-Laplace coefficients (7.3) have the form

$$f_{m,\mu}(r) \stackrel{\text{a.e.}}{=} \begin{cases} 0 & \text{if } m=0, 1, \\ \sinh^{-n} \sum_{j=1}^{[m/2]} c_j \coth^{m-2j} \psi & \text{if } m \geq 2, \end{cases} \quad (7.5)$$

with some constants c_j . Conversely, for any constants c_j and any f satisfying (7.4) and (7.5), we have $(\mathfrak{R}f)(\xi) = 0$ a.e. on \mathbb{H}^n .

8. CONCLUSION

The list of projectively equivalent Radon-like transforms, the kernels of which admit effective characterization using the results of the present paper, can be essentially extended. For example, one can obtain kernel descriptions of the exterior/interior analogues of the Radon-like transforms (and their duals) from Sections 4-7. We leave this useful exercise to the interested reader.

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